CS 6862 Lecture 19, April 5, 2011

Kleene's Theorem on Recursive Fans

Let 2^* be List({0,1}) and $2^{\mathbb{N}}$ be $\mathbb{N} \to \{0,1\}$. For lists l_1, l_2 let $l_1 @ l_2$ be the append operation $[1,2,3] @ [4,5,6] = [1,2,3,4,5,6]$.

Let $GR(\alpha)$ mean that function $\alpha \in \mathbb{N} \to \{0,1\}$ is <u>general recursive</u>. According to the Kleene Normal Form Theorem, this means that we can find an index $a \in \mathbb{N}$ such that $\alpha = \varphi_a$ where φ_i is an enumeration of the partial recursive functions and the T-predicate $T(i,x,z)$ says that z is a terminating computation of $\varphi_i(x)$. The function $U(z)$ picks out the function value from the computation, so

$$\varphi_i(x) = U(\mu z\, T(i,x,z))$$

where μz picks out the least z satisfying the T-predicate. The predicate T and function U are primitive recursive.

Theorem $\exists R: 2^* \to \mathbb{B}$, R primitive recursive, such that

$\forall l_1, l_2 : 2^*.\ R(l_1 @ l_2) \supset R(l_1)$ &

$\forall \alpha : GR(2^{\mathbb{N}}).\ \exists x : \mathbb{N}.\ \neg R(\bar{\alpha}(x))$ & (1)

$\forall x : \mathbb{N}.\ \exists l : 2^*.\ (|l| = x \,\&\, R(l))$. (2)

Note, $|l|$ is the length of list l and $\bar{\alpha}(x)$ is the finite list $[\alpha(0), \alpha(1), \ldots, \alpha(x)]$.

<u>Proof</u>

Define $R(l) == \forall y < |l|.\ [((\exists u < |l|.\ T(y,y,u) \,\&\, U(u) = 0) \supset l(y) = 1)$
$\& ((\exists u < |l|.\ T(y,y,u) \,\&\, U(u) = 1) \supset l(y) = 0)]$

Where $l(y)$ is the y-th element of the list l.

Proof continued.

Notice that $\forall \ell_1, \ell_2 : 2^*.\ R(\ell_1 @ \ell_2) \supset R(\ell_1)$ because if a property holds for all y and u less than $|\ell_1 @ \ell_2|$, then it holds for all y, u less than $|\ell_1|$.

We need to show (1) and (2). We first show (2).

Given any $x : \mathbb{N}$, define the sequence ℓ_x for each $y < x$ as

$$\ell(y) = 1 \text{ iff } \exists u < x.(T(y,y,u) \& U(u) = 0)$$

Clearly $R(\ell)$, that is

$$\forall y < |\ell|.[((\exists u < |\ell|.\ T(y,y,u) \& U(u)=0) \supset \ell(y) = 1) \& ((\exists u < |\ell|.\ T(y,y,u) \& U(u) = 1) \supset \ell(y) = 0)]$$

Now to prove (1) we note that no general recursive α can satisfy R for all x, indeed R fails on the index of α. Suppose $\alpha = \phi_a$, $\phi_a(a)$ has value 0 or 1. Since ϕ_a is total, there is a computation that terminates on input a, say v_a; thus $T(a, a, v_a)$ and either $U(v_a) = 0$ or $U(v_a) = 1$.

If $U(v_a) = 0$ then $\phi_a(a) = 0$ and thus for $x = a + v_a + 1$, $\neg R(\bar{\alpha}(x))$.

If $U(v_a) = 1$ then $\phi_a(a) = 1$ and also $\neg R(\bar{\alpha}(x))$.

Qed.

Cor: Fan_D is false on $2^{\mathbb{N}}$ since $\neg R(x)$ bars the fan by (1) but there are arbitrarily long paths to the bar by (2). Intuitive this must be true since the halting problem $\phi_i(i)\downarrow$ is unsolvable, hence no recursive bound on the length of a terminating computation.